Degenerate vacuum formalism for spontaneous symmetry breakdown

1971 J. Phys. A: Gen. Phys. 4 632

(http://iopscience.iop.org/0022-3689/4/5/004)

J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Degenerate vacuum formalism for spontaneous symmetry breakdown

R. N. SEN and C. WEIL

Department of Physics, Technion-Israel Institute of Technology, Haifa, Israel

MS. received 3rd September 1970, *in revised form* 14*th April* 1971

Abstract. The transformations $\phi(x) \rightarrow \phi(x) + \alpha$ of the free massless scalar field are rendered implementable in a separable Hilbert space by adjoining two conjugate operators (belonging to the zero-energy solutions of the wave equation) to the algebra of the usual particle creation and annihilation operators. The result is an alternative formalism for the discussion of spontaneous symmetry breakdown. It is shown that extension of the formalism to the transformations $A_\mu(x) \rightarrow A_\mu(x) + \alpha_\mu$ of a free massless vector field yields nontrivial zero-energy representations of the Poincaré group.

1. Introduction

In order to formulate the problem which we will discuss in this article, let us briefly recall the essential features of spontaneous symmetry breakdown. Let two scalar Heisenberg fields $\phi_1(x)$ and $\phi_2(x)$ be related by an internal symmetry transformation

$$T^{-1}\phi_1(x)T = \phi_2(x). \tag{1}$$

By definition, the operator T, which is formally unitary, commutes with the generators of the translations:

$$[P_\mu, T] = 0. \tag{2}$$

Then, if the vacuum state $|0\rangle$, conventionally defined by $P_\mu|0\rangle = 0$, has the properties

$$\langle 0|\phi_1|0\rangle = 0 \qquad \langle 0|\phi_2|0\rangle = \chi \neq 0 \tag{3}$$

the symmetry defined by (1) is said to be 'spontaneously broken'. In that case, taking the vacuum expectation value of equation (1) we obtain, using (3), that

$$\langle 0|T^{-1}\phi_1(x)T|0\rangle = \chi \neq 0 \tag{4}$$

which cannot be met if $T|0\rangle = |0\rangle$, as in normal field theories. If on the other hand $T|0\rangle \neq |0\rangle$ then from (2) we find that

$$P_\mu(T|0\rangle) = 0$$

that is, the state $T|0\rangle$, if it exists, is degenerate with the vacuum.

There are thus two possibilities for building up a Hilbert space in which the spontaneous symmetry breakdown conditions (3) can be realized: (i) *The vacuum state is unique*, in which case the operator T cannot be defined on it, and care is required in extracting physical information from the algebra of the symmetry operators. (ii) *The operator T is defined on the vacuum state*, in which case the latter must be degenerate in the physical Hilbert space.

The second of these formulations does not seem to have been developed in the literature†. We propose to fill this gap in the present paper. We will define a separ-

† We are informed by the referee that the contents of section 2 had been developed independently by B. Zumino (unpublished).

able Hilbert space with a degenerate vacuum in which all symmetry operations are unitarily implementable. However, before this can be done, it is necessary to digress on the structure of the theory in the first formulation to track down precisely why the operator T is not implementable in it.

Consider therefore, following Streater (1965), the simple example of a free massless Hermitian scalar field. The field equation

$$\Box\phi(x) = 0 \tag{5}$$

is invariant under the canonical transformations

$$\phi(x) \to \phi(x) + \alpha \tag{6}$$

where α is any real constant. However, these transformations cannot be implemented, that is, there exists no unitary operator satisfying (2) such that

$$T^{-1}(\alpha)\phi(x)T(\alpha) = \phi(x) + \alpha. \tag{7}$$

The argument is the same as in the first paragraph, with $T(\alpha)$, $\phi(x)$ and $\phi(x) + \alpha$ replacing T, $\phi_1(x)$ and $\phi_2(x)$ respectively.

This seemingly trivial model displays all essential features of spontaneous symmetry breakdown in the absence of gauge Heisenberg fields owing to Goldstone's theorem (Goldstone 1961, Goldstone *et al.* 1962) and the dynamical rearrangement of symmetry (Sen and Umezawa 1967). The latter states that whenever a continuous internal symmetry is spontaneously broken (again, in the absence of gauge Heisenberg fields), the corresponding one-parameter transformations of the original Heisenberg fields are rearranged to zero-energy transformations (6) of massless scalar Goldstone fields. In a unique vacuum framework these transformations, as shown above, are not implementable.

The reason why these transformations are not implementable is that the usual Fock space is built out of *finite-energy* solutions of the wave equation (5), whereas the real constants belong to the manifold of *zero-energy* solutions. The transformations (6) change the 'amplitude', in $\phi(x)$, of the zero-energy solution. Indeed, the usual plane wave decomposition of the field

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{2k_0} \{a_k \exp(-\mathrm{i}kx) + a_k^\dagger \exp(\mathrm{i}kx)\}$$

where

$$k_0 = |\boldsymbol{k}| \qquad \text{and} \qquad kx = k_0x_0 - \boldsymbol{k} \cdot \boldsymbol{x}$$

implies, in the limit $\boldsymbol{k} \to 0$

$$a_0 = -a_0^\dagger$$

that is, a_0 and $a_0^\dagger$ cancel each other in $\phi(x)$. Furthermore, they are not canonically conjugate and do not destroy or create zero-momentum particles.

It now becomes obvious that to arrive at a Hilbert space in which the transformations (6) are implementable, one has to include in it the zero-energy solutions of (5), which the Fock prescription does not do.

In the next section we construct the degenerate vacuum appropriate to the Goldstone mechanism. In § 3 we extend our formalism to the zero-energy transformations of a free massless vector field and construct an example of a nonLorentz invariant vacuum. Finally, we make some remarks on the implications of this work.

2. Implementation of the zero-energy transformations of a massless neutral scalar field

In order to safeguard separability, we have to work with wave packets. We use the following notations: let $\{f_i(\boldsymbol{k})\}$ be a countable orthonormal basis in the space $L^2(R^3)$ of square-integrable complex functions over the three-dimensional Euclidean space R^3

$$(f_i, f_j) = \int f_i^*(\boldsymbol{k}) f_j(\boldsymbol{k})\, \mathrm{d}^3k = \delta_{ij}.$$

Define, as usual,

$$u_i(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{(2k_0)^{1/2}} f_i(\boldsymbol{k}) \exp(-\mathrm{i}kx).$$

We now agree to set $f_i(0) = 0$ for all i. This stipulation is Lorentz-invariant in the context which follows. The functions $u_i(x)$ form an orthonormal basis in the space of normalizable positive-energy solutions of equation (5), with the customary Klein–Gordon scalar product

$$(u, v)_{\mathrm{KG}} \equiv \mathrm{i} \int u^*(x) \overleftrightarrow{\partial_0} v(x)\, \mathrm{d}^3x$$

by virtue of the isometry

$$(u_i, u_j)_{\mathrm{KG}} = (f_i, f_j).$$

The usual wave-packet expansion of $\phi(x)$ does not include a zero-energy part. We denote the zero-energy part by A and write the most general expansion as

$$\phi(x) = A + \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)] \tag{8}$$

where A is a Hermitian operator, independent of x.

The transformations (6) can now be isolated in the form

$$T^{-1}(\alpha) A T(\alpha) = A + \alpha.$$

If the operators $T(\alpha)$ are defined on a Hilbert space, they form a one-parameter group of unitary transformations. Hence there exists a Hermitian operator Q such that

$$T(\alpha) = \exp(-\mathrm{i}\alpha Q)$$

with

$$[A, Q] = \mathrm{i}. \tag{9}$$

A and Q commute with all a_i and $a^\dagger$.

Thus, in a Hilbert space in which the transformations (6) are implementable, the algebra of the 'particle operators' a_i and $a_i^\dagger$ is augmented by the 'vacuum operators' A and Q. The particle operators by themselves do not form an irreducible set. This is the essential new step which is required. The rest follows automatically.

For example, in a 'measurable field' interpretation of the degenerate vacuum we would want the states of the system to approximate to eigenstates of A:

$$A|\omega\rangle = \omega|\omega\rangle$$
$$\langle\omega|\omega'\rangle = \delta(\omega - \omega')$$

where ω is any real number. Then

$$T(\alpha)|\omega\rangle = |\omega+\alpha\rangle.$$

Let $\{\eta_i(\omega)\}$ be a countable orthonormal basis of functions of fast decrease in $L^2(R)$, and define a countable orthonormal basis of states by

$$|i\rangle = \int \mathrm{d}\omega\eta_i(\omega)|\omega\rangle.$$

The matrix elements of A and $T(\alpha)$ are now

$$\langle i|A|j\rangle = \int \omega\eta_i^*(\omega)\,\eta_j(\omega)\,\mathrm{d}\omega$$

and

$$\langle i|T(\alpha)|j\rangle = \int \eta_i^*(\omega)\eta_j(\omega-\alpha)\,\mathrm{d}\omega.$$

The states $|i\rangle$ are vacuum states, that is

$$P_\mu|i\rangle = 0 \qquad \text{for all } i$$

and the orthonormal n-particle states are constructed as usual:

$$|i; n_1 \ldots n_k \ldots\rangle = (n_1! \ldots n_k! \ldots)^{-1/2}(a_1^\dagger)^{n_1} \ldots (a_k^\dagger)^{n_k} \ldots |i; 0\rangle$$
$$n \equiv \sum n_k < \infty$$

where we have written $|i\rangle$ as $|i; 0\rangle$ for clarity. This Hilbert space is a cyclic representation of the augmented algebra based on the operators A, Q, $a^\dagger$ and a_i.

An alternative form of the degenerate vacuum can be constructed with a 'spurion' interpretation in mind. Define the spurion creation and annihilation operators

$$s = \frac{1}{\sqrt{2}}(A+\mathrm{i}Q) \qquad s^\dagger = \frac{1}{\sqrt{2}}(A-\mathrm{i}Q)$$
$$[s, s^\dagger] = 1$$

and construct the degenerate vacua $|\nu\rangle$ as eigenstates of the spurion number operator $s^\dagger s$. It is worth repeating that these spurions have no relation to the $\boldsymbol{k} \to 0$ limit of massless quanta. In this representation, the states with no spurions form the usual Fock space.

3. Implementation of the zero-energy transformations of a massless vector field

The considerations of § 2 can easily be extended to the transformations

$$a_\mu(x) \to a_\mu(x)+\alpha_\mu \qquad \alpha_\mu \text{ constant} \tag{10}$$

of a massless vector field $a_\mu(x)$. One arrives at degenerate vacua which are unitary representations of type 3 (Wigner 1939) of the Poincaré group, that is, infinite-dimensional unitary representations of the Lorentz group.

As before, we write the zero-energy parts of $a_\mu(x)$ as A_μ, which are now Hermitian operators, and introduce the operators Q_μ which are conjugate to them:

$$[Q_\mu, A_\nu] = \mathrm{i}g_{\mu\nu}.$$

Define the vacuum states $|\omega\rangle = |\omega_0, \omega_1, \omega_2, \omega_3\rangle$ to be the eigenstates of A_μ in the continuum normalization:

$$A_\mu|\omega\rangle = \omega_\mu|\omega\rangle$$
$$\langle\omega|\omega'\rangle = \delta^4(\omega-\omega').$$

The Lorentz-transformation properties of the states $|\omega\rangle$ follow from the observation that A_μ, being the zero-energy part of $a_\mu(x)$, is a four-vector, and so is Q_μ. If we assume that A_μ and Q_μ together form an irreducible set of zero-energy operators, it is easy to see that the antisymmetric Hermitian tensor operators

$$M_{\mu\nu} \equiv A_\mu Q_\nu - A_\nu Q_\mu$$

generate the Lorentz transformations on the states $|\omega\rangle$. In fact, the Lorentz transformation Λ which takes A to $A' = \Lambda A$ $(= \Lambda_\mu{}^\nu A_\nu)$ takes $|\omega\rangle$ to $|\Lambda^{-1}\omega\rangle$. Moreover, it is clear from (8) and (10) that A_μ is invariant under the translations

$$a_\mu(x) \to a_\mu(x+y).$$

Thus the infinite-dimensional manifold of states $|\Lambda\bar{\omega}\rangle$, where Λ is a Lorentz transformation and $\bar{\omega}$ a fixed four-vector, forms an irreducible unitary representation of the Lorentz group (which is also a type 3 representation of the Poincaré group) on a nonseparable Hilbert space (the unitarity may be verified explicitly by calculating the matrix elements of $M_{\mu\nu}$). Separability may be retrieved trivially, as before, by constructing 'wave packets'.

As in § 2, we can build the degenerate vacuum in a spurion interpretation. We define

$$s_\mu = \frac{1}{\sqrt{2}}(A_\mu + \mathrm{i}Q_\mu)$$
$$s_\mu^\dagger = \frac{1}{\sqrt{2}}(A_\mu - \mathrm{i}Q_\mu).$$

Then

$$[s_\mu, s_\nu^\dagger] = -g_{\mu\nu}$$
$$[s_\mu, s_\nu] = [s_\mu^\dagger, s_\nu^\dagger] = 0.$$

The 'wrong' sign of $[s_0, s_0^\dagger]$ plays an interesting role. As required by it, we interpret $s_0^\dagger$ as an annihilation operator and construct the Fock space of spurions in the usual manner. Owing to the absence of the subsidiary condition, there is no restriction on admissible state vectors, and no vectors which cannot be normalized appear (Jauch and Rohrlich 1955, pp. 100–2). Thus the indefinite metric is not required. Now denote s_0 by $b^\dagger$ and $s_0^\dagger$ by b, and write explicitly

$$M_{ij} = -\mathrm{i}(s_i^\dagger s_j - s_j^\dagger s_i)$$
$$M_{i0} = \mathrm{i}(s_i b - s_i^\dagger b^\dagger).$$

The last expression makes it evident that no finite-dimensional sector of the Fock space of spurions is Lorentz-invariant; there are no nontrivial finite-dimensional unitary representations of a noncompact group. The Lorentz-invariant spaces are characterized by a constant *difference* between the number of 'spacelike' and 'timelike' spurions.

4. Concluding remarks

(i) We have shown that the theory of degenerate vacua associated with implementing the zero-energy transformations of the massless scalar field can be developed by elementary means. It is evident that the formalism can be extended to the dynamical rearrangement of symmetry, with a certain gain in clarity owing to the disappearance of non-implementable operators from the theory.

(ii) The Lagrangian formulation of field theory cannot be extended to incorporate implementable zero-energy transformations. This is because the canonical definition of conjugate momentum

$$\Pi(x) = \frac{\partial L}{\partial \dot{\phi}(x)}$$

does not admit a zero energy part in $\Pi(x)$, even if the massless field $\phi(x)$ contains such a part.

(iii) The method outlined in § 3 applies equally to massless tensor fields of higher rank, and to indefinite metrics other than the Minkowski one. It provides an elementary means of *calculating explicitly*, in the spurion picture, a class of nontrivial irreducible unitary representations of noncompact groups.

References

Goldstone, J., 1961, *Nuovo Cim.*, **19**, 154–64.

Goldstone, J., Salam, A., and Weinberg, S., 1962, *Phys. Rev.*, **127**, 965–70.

Jauch, J. M., and Rohrlich, F., 1955, *Theory of Photons and Electrons* (Reading, Mass: Addison-Wesley).

Sen, R. N., and Umezawa, H., 1967, *Nuovo Cim.*, **50**, 53–74.

Streater, R. F., 1965, *Proc. R. Soc.* A, **287**, 510–8.

Wigner, E. P., 1939, *Ann. Math.*, **40**, 149–204; Reprinted 1966, in *Symmetry Groups in Nuclear and Particle Physics*, ed. F. J. Dyson (New York: Benjamin).